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THE CONDITIONS FOR THE SYMMETRIC INSTABILITY OF THE VORTEX MOTIONS OF AN IDEAL STRATIFIED LIQUID†

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The problem of the symmetric instability of the steady-state motions of an incompressible ideal liquid which is stratified with respect to its density is investigated in the case of two types of motion, axially symmetric and with translational symmetry. It is shown that the sufficient condition for stability obtained in [1] using a variational method (the direct Lyapunov method) for the motions under consideration is closely related to the extremal nature of their energy; stable motions are characterized by a conditional minimum of the energy. A minimum of the energy holds in the class of states for which a potential vortex, expressed in terms of the Lagrangian invariants, angular momentum and density, is represented by the same function of these invariants. Conditions for instability are formulated and estimates of the increase in the kinetic energy of perturbations are given.

1. Consider the axially symmetric motion of an ideal liquid, which is stratified with respect to its density, in the incompressibility approximation. The system of dynamic equations in a cylindrical system of coordinates (r, φ, z) with a vertical z-axis has the form

$$\begin{aligned} &\frac{du}{dt}-\frac{M^2}{r^3}=-\frac{1}{\rho}\frac{\partial p}{\partial r},\ \frac{dw}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g \\ &\frac{dM}{dt}=0,\ \frac{d\rho}{dt}=0,\ \frac{1}{r}\frac{\partial ru}{\partial r}+\frac{\partial w}{\partial z}=0 \end{aligned} \tag{1.1}$$

Here $M = r\upsilon$ is the angular momentum, u, w, υ are the radial, vertical and azimuthal components of the velocity, ρ is the density, p is the pressure and g is the acceleration due to gravity. The liquid occupies the cylindrical domain: $0 \leqslant r \leqslant R, 0 \leqslant z \leqslant H$, on the boundaries of which the normal component of the velocity vanishes. We shall assume that the Jacobian $D(M, \rho)/D(r, z)$ does not identically vanish anywhere in the above domain. The steady state (the cyclostrophic balance), which is denoted by the subscript s, is then described by the equations

$$\begin{aligned} &u_s = w_s = 0;\ \rho_s\frac{M_s^2}{r^3}=\frac{\partial p_s}{\partial r} \\ &-g\rho_s=\frac{\partial p_s}{\partial z} \end{aligned} \tag{1.2}$$

A sufficient condition for the symmetric stability of this state, obtained by the direct Lyapunov method [1], is formulated in the following manner: if the inequalities

$$\frac{\partial \rho_s}{\partial z}<0,\ D(M_s^2,\rho_s)/D(r,z)<0$$

are satisfied over the whole of the flow domain then the state (1.2) is stable in a Lyapunov sense in theclass of axially symmetric perturbations. This condition can be written, using Eqs (1.2), in the equivalent form

$$\frac{\partial^2 p_s}{\partial z^2}>0,\ \frac{\partial^2 p_s}{\partial a^2}\frac{\partial^2 p_s}{\partial z^2}-\left(\frac{\partial^2 p_s}{\partial a\partial z}\right)^2>0 \tag{1.3}$$

where $a = r^{-2}$.

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The physical meaning of this condition consists of the fact that the energy is a minimum in the steady state.

We shall now refine and substantiate what has been said above. We denote ρM^2 by μ. It is clear that μ is a Lagrangian invariant: $d\mu/dt = 0$. Furthermore, the expression $\Omega = r^{-1}D(\mu, \rho)/D(r, z)$ is also a Lagrangian invariant (in the appropriate variables, it is proportional to the Ertel potential vortex).

It is fairly obvious that, on account of the two-dimensional nature of the dynamics, a functional relation of the type $\Omega = F(\mu, \rho)$ must exist, at least locally (this has been called [2] a functional invariant). If the Jacobian $D(\mu, \rho)/D(r, z)$ does not vanish over the whole of the flow domain, this relation also exists globally. In this case, the set of all states of axially symmetric motion which is characterized by the collection (u, w, μ, ρ) naturally decomposes into equivalence classes: identical functional invariants correspond to equivalent states. In particular, orbits (that is, states associated with the dynamics) consist of equivalent states.

The assertion mentioned above is formulated as follows: when the stability condition (1.3) is satisfied, the energy of the steady state has an absolute minimum in the class of equivalent states.

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We will now indicate how this is proved. Using the coordinates $a = r^{-2}$, z and the function p_s, we define the Legendre transformation

$$\alpha = \partial p_s/\partial a, \ \beta = \partial p_s/\partial z \tag{1.4}$$

The function p_s which is dual to $\phi(\alpha, \beta) = \alpha a + \beta z - p_s$ has the derivatives: $\partial\phi/\partial\alpha = a$; $\partial\phi/\partial\beta = z$ when $(\alpha, \beta) = (-\mu_s/2, g\rho_s)$ by virtue of (1.4) and (1.2), and also satisfies inequalities which are similar to (1.3)

$$\frac{\partial^2\phi}{\partial\alpha^2} > 0, \ \frac{\partial^2\phi}{\partial\alpha^2}\frac{\partial^2\phi}{\partial\beta^2} - \left(\frac{\partial^2\phi}{\partial\alpha\partial\beta}\right)^2 > 0 \tag{1.5}$$

since it is a property of a Legendre transformation that convexity is preserved. The inequality

$$\phi(\alpha,\beta) - \phi(-\frac{\mu_s}{2}, \ -g\rho_s) - (\alpha + \frac{\mu_s}{2})a - (\beta + g\rho_s)z > 0 \tag{1.6}$$

when $(\alpha, \beta) \neq (-\mu_s/2, g\rho_s)$, follows from this.

Actually, the expression on the left-hand side, as a function of α, β, vanishes together with its first derivatives when $(\alpha, \beta) = (-\mu_s/2, g\rho_s)$, and we establish inequality (1.6) when account is taken of the convexity of $\phi(\alpha, \beta)$.

The difference in the energies of a certain state (u, w, μ, ρ) and the steady state $(0, 0, \mu_s, \rho_s)$, which is equivalent to it, can be represented in the form

$$\Delta E = 2\pi\int_0^R rdr\int_0^H dz\left[\frac{\rho u^2}{2} + \frac{\rho w^2}{2} + \frac{(\mu-\mu_s)a}{2} + (\rho-\rho_s)gz + \right.$$
$$\left. + \left\{\phi(-\frac{\mu}{2}, -g\rho) - \phi(-\frac{\mu_s}{2}, -g\rho_s)\right\}\right]$$

The integral of the expression in the braces vanishes, which is proved by representing it in the form of a difference of integrals and making the change of integration variables $(r, z) \to (\mu, \rho)$, $(r, z) \to (\mu_s, \rho_s)$, taking into account the same expression for the Jacobians which follows from the definition of equivalence.

Using inequality (1.6), we conclude that $\Delta E > 0$ when $(u, w, \mu, \rho) \neq (0, 0, \mu_s, \rho_s)$, which it was required to prove.

2. Suppose that at least one of the conditions

$$\frac{\partial^2 p_s}{\partial z^2} < 0, \ \frac{\partial^2 p_s}{\partial a^2}\cdot\frac{\partial^2 p_s}{\partial z^2} - \left(\frac{\partial^2 p_s}{\partial a\partial z}\right) < 0 \tag{2.1}$$

holds in a certain domain.

We shall show that the steady state is then unstable. In order to do this, as is well known, it is sufficient to demonstrate the instability of the dynamic equations in the neighbourhood of the steady state [3]. As noted in [4], it is convenient to use the Lagrange approach here. The linearized displacement field $\xi(r, z, t)$, $\eta(r, z, t)$ is defined by the relations

$$\xi=\xi_0(r,z)+\int_0^t u(r,z,t')dt' \tag{2.2}$$

$$\eta=\eta_0(r,z)+\int_0^t w(r,z,t')dt'$$

and satisfies the linearized incompressibility equation

$$\frac{1}{r}\frac{\partial r\xi}{\partial r}+\frac{\partial\eta}{\partial z}=0$$

and, also, the boundary conditions

$$\xi(R, z) = \eta(r, 0) = \eta(r, H) = 0.$$

It is assumed that the initial displacements ξ_0, η_0 also satisfy these conditions. We shall confine ourselves to variations in μ and ρ, which are defined by the equalities

$$\delta\rho=-\left(\xi\frac{\partial}{\partial r}+\eta\frac{\partial}{\partial z}\right)\rho_s$$

$$\delta\mu=-\left(\xi\frac{\partial}{\partial r}+\eta\frac{\partial}{\partial z}\right)\mu_s \tag{2.3}$$

It can be shown that the perturbations $\mu' = \mu_s + \delta\mu$, $\rho' = \rho_s + \delta\rho$ are equivalent (in the linear approximation) to the steady state. Physically, such a perturbation at the initial instant of time is obtained by the displacement of the particles from their equilibrium positions while preserving the elementary volumes and the corresponding values of ρ_s and μ_s.

The linearized system (1.1) in the variables ξ, η reduces to the form

$$\rho_s\ddot{\xi}+\xi A+\eta B=-\frac{\partial p'}{\partial r}$$

$$\rho_s\ddot{\eta}+\xi B+\eta C=-\frac{\partial p'}{\partial z} \tag{2.4}$$

$$\left(A=\frac{1}{r^3}\frac{\partial\mu_s}{\partial r},\ B=-g\frac{\partial\rho_s}{\partial r}=\frac{1}{r^3}\frac{\partial\mu_s}{\partial z},\ C=-g\frac{\partial\rho_s}{\partial z}\right)$$

where p' are the pressure perturbations.

Conditions (2.1) mean that the matrix

$$\hat{F}=\begin{Vmatrix} A & B \\ B & C \end{Vmatrix} \tag{2.5}$$

has at least a single negative eigenvalue.

System (2.4) has a quadratic energy integral $E = T + U$, where

$$T=\frac{1}{2}\int_0^R rdr\int_0^H dz\rho_s(\dot{\xi}^2+\dot{\eta}^2)$$

$$U=\frac{1}{2}\int_0^R rdr\int_0^H dz(\xi^2A+2\xi\eta B+\eta^2C) \tag{2.6}$$

It can be shown (see the Appendix) that, when at least one of conditions (2.1) is satisfied, there is an initial displacement field to which a negative value of U corresponds.

To prove instability we use the functional [4]

$$W = \frac{1}{2}\int_0^R dr r \int_0^H dz \rho_s(\xi^2 + \eta^2) \tag{2.7}$$

Its time derivative satisfies the inequality $\dot{W}^2 \leqslant 4TW$ (which is proved using the Cauchy–Bunyakovskii inequality).It follows from the equation of motion that $\ddot{W} = 4T - 2E$. From these relations, we obtain the inequality

$$\frac{d}{dt}\left(\frac{\dot{W}}{W}\right) \geq -\frac{2E}{W} \tag{2.8}$$

We select the initial displacement field (ξ_0, η_0) such that it corresponds to $U(0) < 0$ and put the velocities $\dot{\xi}(0)$, $\dot{\eta}(0)$ equal to $\varkappa \cdot (\xi_0, \eta_0)$, where we select the constant $\varkappa > 0$ from the condition that $E = 0$. From this, we obtain that $\varkappa = (|U(0)|/W(0))^{1/2}$. In this case, it is obvious that the equality $\dot{W}(0) = 2\varkappa W(0)$ is satisfied. For the given initial condition, we obtain the estimates

$$\frac{\dot{W}}{W} \geq 2\varkappa; \; W(t) \geq W(0)\exp(2\varkappa t)$$

from inequality (2.8).

The value of the constant $\varkappa$ obviously defines the lower boundary of the increment in the growth of the perturbations.

Correspondingly, for the kinetic energy, we have the estimate

$$T \geq W(\dot{W}/W)^2/4 \geq \varkappa^2 W(t) \tag{2.9}$$

We therefore obtain an exponential estimate of the growth in the kinetic energy of a perturbation. On the basis of the foregoing discussion it can be shown that, in the final analysis, instability of the steady state is due to the existence of equivalent perturbations with energy values which are smaller than in the case of the steady state. States which correspond to perturbations (2.3) with a displacement field for which $U(0) < 0$ and $T(0) = 0$ are examples of such states.

3. The issue on the instability of a steady state of the geostrophic balance, for which the pressure gradient is balanced by the Coriolis force [5], can be investigated in a similar manner. Most frequently, a model of the flow in which the fields are independent of one of the horizontal coordinates is used. Suppose that this is the y axis. Then, the dynamic equations in a Cartesian system of coordinates can be written in the form [5]

$$\begin{aligned} &\frac{du_x}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fu_y, \; \frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \\ &\frac{du_y}{dt} = -fu_x, \; \frac{\partial \rho}{\partial t} = 0 \\ &\frac{du_x}{dx} + \frac{\partial w}{\partial z} = 0 \; \left(\frac{d}{dt} = \frac{\partial}{\partial t} + u_x\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}\right) \end{aligned} \tag{3.1}$$

where u_x, u_y are the components of the velocity along the x and y axes, and f is the Coriolis parameter. The third equation of (3.1) means that the quantity $m = u_y + f_x$ is a Lagrangian invariant: $dm/dt = 0$. This so-called geostrophic momentum is, to a known extent, analogous to the angular momentum M. Motion is considered in the domain $0 \leqslant x \leqslant l$, $0 \leqslant z \leqslant H$, on the boundaries of which the normal component of the velocity vanishes. We assume that the Jacobian $D(m, \rho)/D(x, z)$ is not identically equal to zero. In this case, the steady state (the geostrophic balance) satisfies the equations

$$\begin{aligned} &w_s = 0, \; u_{xs} = 0, \; \frac{\partial p_s}{\partial z} = -g\rho_s \\ &\frac{\partial p_s}{\partial x} = f(m_s - fx)\rho_s \end{aligned} \tag{3.2}$$

A sufficient condition for the symmetric stability of state (3.2) is the following condition [1]: when the conditions

$$\frac{\partial \rho_s}{\partial z} < 0, \quad \frac{fD(m_s, \rho_s)}{D(x,z)} < 0 \tag{3.3}$$

are satisfied, state (3.2) is stable in the Lyapunov sense in the class of symmetric (independent of y) perturbations. By making the change of coordinates $(x, z) \to (x, q)$, where $q = z + f^2x^2/2g$, and using the steady-state equations, we can represent the stability conditions in the form

$$\frac{\partial^2 p_s}{\partial z^2} > 0, \quad \frac{\partial^2 p_s}{\partial x^2}\frac{\partial^2 p_s}{\partial q^2} - \left(\frac{\partial^2 p_s}{\partial x \partial q}\right)^2 > 0 \tag{3.4}$$

which indicates the convexity of p_s as functions of (x, q).

Next, instead of m, we use the Lagrangian invariant $\mu_g = \rho m$. As in the preceding case, it can be shown that a functional relation of the form

$$\frac{D(\mu_g, \rho)}{D(x,z)} = F(\mu_g, \rho)$$

exists, which is explicitly independent of time if μ_g and ρ are Lagrangian invariants (a functional invariant [2]).

The assertion regarding the minimum energy of the steady state holds: when the stability conditions are satisfied, the energy of the steady state has the absolute minimum among states with the specified functional invariant (an energy which is calculated per unit length of y is implied here). The proof uses a Legendre transformation with the function $p_s(x, q)$ and is identical with the preceding proof.

Conditions of instability are formulated in the following way: if, in a certain domain, at least one of inequalities (3.4) changes sign, the state of the cyclostrophic balance is unstable.

The proof is carried out using the scheme employed in the previous case. As before, the instability of the steady state is due to the existence of equivalent states with a lower value of the energy.

4. *Appendix.* We will now outline a scheme for proving the assertion: if the matrix (2.5) with continuous coefficients has a negative eigenvalue in a certain domain, then a displacement field ξ_0, η_0 exists, which satisfies the equation of continuity, for which $U(0) < 0$. We select a point (r_*, z_*) in this domain with a certain ε-neighbourhood which is sufficiently small so that A, B and C only change slightly in it.

We put

$$(\xi_0, \eta_0) = \left(\frac{1}{r}\frac{\partial \psi_0}{\partial z},\ -\frac{1}{r}\frac{\partial \psi_0}{\partial r}\right)$$

and choose the function ψ_0 in the form of a Gaussian distribution with its centre at (r_*, z_*) and positive definite covariant matrix $\hat{\sigma}$ with elements $\sigma_{ik} \sim \varepsilon^2$. Asymptotically, it is possible to obtain: $U(0) = G\mathrm{Sp}(\hat{F}\hat{\sigma})(1 + o(1))$, where $G > 0$. It is also possible to write: $U(0) \sim G(\lambda_1\sigma'_{11} + \lambda_2\sigma'_{22}$, where λ_i is an eigenvalue of the matrix $\hat{F}$ of (2.5), $\hat{\sigma}' = \hat{D}\hat{\sigma}\hat{D}^{-1}$ and the matrix $\hat{D}$ reduces (2.5) to diagonal form. Suppose, for example, that $\lambda_1 < 0$. On choosing a covariant matrix which satisfies the conditions $\sigma'_{11}|\lambda_1| > \lambda_2\sigma'_{22}$, we obtain the required assertion.

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